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Holography at the U.S. Army Research Laboratory: Development of Hologram Transform Equations using the Fresnel Approximation

by Karl K. Klett, Jr., Neal Bambha, and Justin Bickford

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Sensors and Electron Devices Directorate, ARL

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14. ABSTRACT

This report describes the development and use of the Fresnel approximation, which is necessary to transform a digital hologram to an image. The result is that a Fourier transform, common to many image processing applications, may be used to produce the image. The U.S. Army Research Laboratory's (ARL) goal is to investigate applications of holographic interferometry and transform the holographic interference pattern to three-dimensional images. The mathematics described in this report are a critical step in this process. Stokes's theorem is used as a starting point, where the spherical wave complex amplitude is used in the evaluation. The final result of this first step is the Helmholtz-Kirchhoff equation. Assumptions are made, recognizing that only the illuminated information in the aperture is important to the problem. These assumptions change the Helmholtz-Kirchhoff equation to the Kirchhoff diffraction integral. Finally, the Fresnel and Fraunhofer approximations, which deal with aperture coordinates, and aperture size and distance, respectively, are used to obtain a practical solution. The result of these calculations is that the Fourier transform may be used to form an image, instead of more cumbersome integral equations. This fact makes the transformation from a hologram interference pattern to an image much easier to calculate.

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1. Introduction

The U.S. Army Research Laboratory is embarking on a program of holographic interferometry to understand its limitations for the purpose of remote sensing. A pre-requisite for this work is to record a digital hologram. When two holograms are recorded and processed, using two different wavelengths, three-dimensional images are formed. Such images, depending on the wavelengths selected, show depth information that can approach microscopic dimensions. Other advantages of using holographic techniques are the large depth of field, the lack of a need for mechanical focusing mechanisms, and the perfect image reconstruction that possess both phase and amplitude information of the object being examined, instead of just the intensity information that is in a regular photograph (*1*).

Although this work can and has been performed with chemical films, digital capture and processing of images can be accomplished much more quickly. For this reason, mathematical methods that can be applied digitally have been developed to process the hologram interference patterns into images. Of particular interest are methods that use Fourier transforms, because this type of transform is well known and has been implemented in many image processing programs, such as IDL and MATLAB, just to name a few.

2. Methods, Assumptions, and Procedure

2.1 Mathematical Analysis, Part 1—Establishing a Mathematical Framework: Developing the Helmholtz-Kirchhoff Equation

The purpose of the following analysis is to develop a method to transform a hologram, which is an interference pattern of light obtained in the laboratory, into an image. There are two main ways to do this. The first method, which is described below, uses a Fourier transform. The second method, not discussed here, uses convolution. We chose to use the Fourier transform, because it has been optimized in many types of computer applications that we use, and because signal processing hardware exists that can optimize the process of performing the transform. We show and discuss this analysis from a pedagogical point of view, making the connecting steps in the analysis clear and discussing what the final results of the analysis mean.

The analysis begins with Stoke’s theorem equation 1, which relates a volume integral to a surface integral. Figure 1 shows the geometry associated with equation 1.

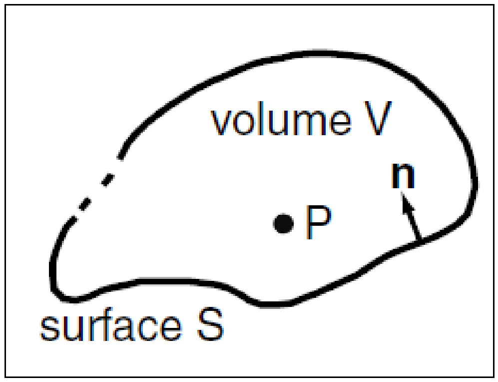

Figure 1. The surface, volume, and normal vector that relate the two integrals in Stoke's theorem.

$$\iiint_V (U_1\Delta^2 U_2 - U_2\Delta^2 U_1)dv = \iint_S (U_1\Delta U_2 - U_2\Delta U_1)\bullet n \;\; ds \tag{1}$$

In this case, the functions U_1 and U_2 in equation 1 are light waves. This means that these functions are solutions to the wave equation. Maxwell's equations show that the volume integral on the left hand side of equation 1 goes to zero. Equation 1 then reduces to

$$0 = \iint_S (U_1\Delta U_2 - U_2\Delta U_1)\bullet n \; ds \tag{2}$$

An analogy with the scaler dot product $(A \bullet B)$ is useful in further evaluating equation 2. The vector product of ***A*** and ***B*** means that the component of vector ***A*** is projected in the ***B*** direction. $(\Delta U \bullet n)$ is therefore the change in U_i in the *n* direction, so equation 2 can be written as

$$0 = \iint_S (U_1 \frac{\partial U_2}{\partial n} - U_2 \frac{\partial U_1}{\partial n})\, ds \tag{3}$$

Consider now figure 2 to proceed in the analysis.

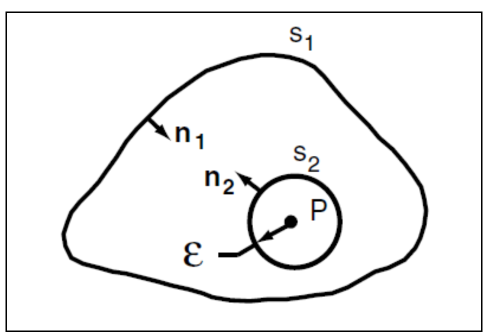

Figure 2. Two surfaces, one of which surrounds a point, used to evaluate equation 3.

The complex amplitude on S_1 is used to calculate the complex amplitude on S_2 surrounding the small sphere, P. The unknown wavefront at S_2, which is a complex amplitude, is evaluated as a spherical wavefront. This analysis works well for holography, where the individual point "P" might be a pixel in a charge-coupled device (CCD) array or the emulsion of a film. The unknown wave at S_2 has the form

$$U_2 = \frac{e^{ikr_2}}{r_2} \tag{4}$$

where "r_2" is the distance from point P, and "k" is the wave number. The integral is now evaluated over each surface using equation 4, where r_2 becomes infinitesimally small. Substitution of equation 4 into equation 3 over S_1 becomes

$$0 = \iint_{S_1} (U_1 \frac{\partial}{\partial n_1}\left(\frac{e^{ikr_2}}{r_2}\right) - \left(\frac{e^{ikr_2}}{r_2}\right)\frac{\partial U_1}{\partial n_1})\, dS_1 \tag{5}$$

The surface integral at surface S_2 is evaluated using the fact that the radius r_2 is in the same direction as n_2.

$$\frac{\partial}{\partial n_2} = \frac{\partial}{\partial r_2} \tag{6}$$

After differentiating, the surface integral for surface S_2 becomes

$$0 = \iint_{S_2} (U_1\left(\frac{ik}{r_2}e^{ikr_2} - \frac{1}{r_2^2}e^{ikr_2}\right) - \frac{e^{ikr_2}}{r_2^2}\frac{\partial U_1}{\partial n_2}) dS_2 \tag{7}$$

Since r_2 (from figure 2) is reduced to zero, $r_2=\varepsilon$, and $dS_2=\varepsilon^2 d\Omega$, equation reduces to

$$0 = \int_0^{4\pi} (U_1 e^{ik\varepsilon}\left(\frac{ik}{\varepsilon} - \frac{1}{\varepsilon^2}\right) - \frac{e^{ik\varepsilon}}{\varepsilon^2}\frac{\partial U_1}{\partial n_2})\varepsilon^2 d\Omega \tag{8}$$

which, when integrated, yields

$$0 = \underbrace{-4\pi\varepsilon^2 U_1 \frac{e^{ik\varepsilon}}{\varepsilon}\left(ik\right.}_{\text{Term 1}} \underbrace{\left. - \frac{1}{\varepsilon}\right)}_{\text{Term 2}} + \underbrace{\varepsilon \int_0^{4\pi} e^{ik\varepsilon}\frac{\partial U_1}{\partial n_2} d\Omega}_{\text{Term 3}} \tag{9}$$

Terms 1 and 3 in equation 9 go to zero as ε goes to zero. Term 2 and equation 9 then reduce to $4\pi U_1$ as ε goes to zero.

Since the complex amplitudes on the two surfaces (S_1 and S_2 in figure 2) are equal, equation 5 and equation 7 are equal. Equation 7 reduces to $4\pi U_1$, as discussed previously, so the equality between S_1 and S_2 becomes

$$U_P = \frac{1}{4\pi}\iint_S \left[U_1 \frac{\partial}{\partial n}\left(\frac{e^{ikr_2}}{r_2}\right) - \frac{e^{ikr_2}}{r_2}\frac{\partial U_1}{\partial n}\right] ds \tag{10}$$

For the digital holography application being considered, think of U_P as being the amplitude at the pixel of a CCD. This result is called the Helmholtz-Kirchhoff equation. Equation 10 is difficult to solve analytically and time consuming if approached computationally.

2.2 Mathematical Analysis, Part 2—Integrating an Aperture Instead of a Full Sphere: The Kirchhoff Assumptions

Surface S_1 of figure 2 is a closed surface encompassing a full hemisphere. The Kirchhoff assumption is that the full sphere is not illuminated; only a small portion (an aperture) of that sphere is illuminated. The following assumptions then apply:

$$U_1 = 0 \;\; and \;\; \frac{\partial U_1}{\partial n} = 0 \tag{11}$$

on the surface S_1 except at the aperture. The following geometry is useful for visualizing the situation and understanding trigonometric relations used in the approximation.

The aperture, which replaces S_1, is approximated as planar because it is such a small portion of the surface S_1. Using the coordinates in figure 3, the spherical wavefront of the source is

$$U_1 = \frac{Ae^{ikr_1}}{r_1} \tag{12}$$

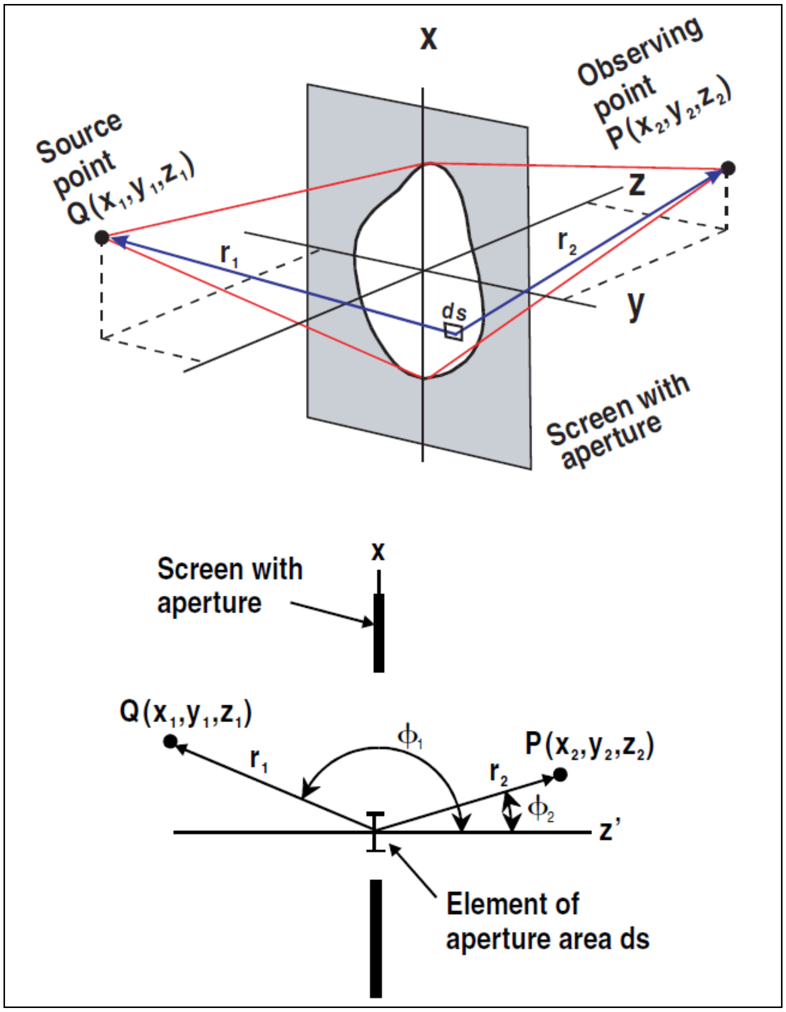

Figure 3. Geometries of the aperture that replaces S_1.

The normal to the aperture does not change (as it does on the surface S_1) and is the z-axis, so equation 12 can be differentiated as follows:

$$\frac{\partial U_1}{\partial n} = \frac{\partial U_1}{\partial z} = A\frac{e^{ikr_1}}{r_1}\left(ik - \frac{1}{r_1}\right)\cos\varphi_1 \tag{13}$$

Light is coming to the aperture from the source of light and from the observing point, which is the object that is illuminated to create a hologram. Light from both of these sources reflect back and interfere at the aperture. The light from the observing point is U_2 and

$$U_2 = \frac{Ae^{ikr_2}}{r_2} \qquad\qquad \frac{\partial U_2}{\partial z} = A\frac{e^{ikr_2}}{r_2}\left(ik - \frac{1}{r_2}\right)\cos\varphi_2 \tag{14}$$

A final assumption is made, and that is that the source and object are more than 1 cm from the aperture, since k=2π/λ~100,000 for visible light, and in equations 13 and 14, if r_1 and r_2 are on the order of 1 cm or greater, then $1/r_1$ and $1/r_2$ can be neglected with respect to k. This means that

$$\frac{1}{r_1} \sim \frac{1}{r_2} << k \tag{15}$$

Equation 10, which is copied below as equation 15, is the starting point for applying the aperture assumptions.

$$U_P = \frac{1}{4\pi}\iint_S \left[U_1 \underbrace{\frac{\partial}{\partial n}\left(\frac{e^{ikr_2}}{r_2}\right)}_{(14)} - \frac{e^{ikr_2}}{r_2}\frac{\partial U_1}{\partial n}\right] ds \tag{16}$$

$$U_P = \frac{1}{4\pi}\iint_{Aperture} \left[\overset{(12)}{\frac{Ae^{ikr_1}}{r_1}} \overbrace{\frac{e^{ikr_2}}{r_2}\left(ik - \cancelto{0 \text{ from } (15)}{\frac{1}{r_2}}\right)\cos\varphi_2}^{(14)} \right] - \left[\frac{e^{ikr_2}}{r_2} \overbrace{A\frac{e^{ikr_1}}{r_1}\left(ik - \cancelto{0 \text{ from } (15)}{\frac{1}{r_1}}\right)\cos\varphi_1}^{(13)} \right] dxdy \tag{17}$$

The numbers in parenthesis are the equations are used to get from equation 16 to equation 17. Combining like terms reduced equation 17 to

$$U_P = \frac{-Aik}{4\pi}\iint_{Aperture} \frac{1}{r_1 r_2} e^{ik(r_1+r_2)}\left(\cos\varphi_1 - \cos\varphi_2\right) dxdy \tag{18}$$

Since k=2π/λ, equation 18 can be further reduced to

$$U_P = \frac{-Ai}{2\pi}\iint_{Aperture} \frac{1}{r_1 r_2} e^{ik(r_1+r_2)}\left(\cos\varphi_1 - \cos\varphi_2\right) dxdy \tag{19}$$

This result is called the Kirchhoff diffraction integral and is still difficult to integrate in most situations. The reason is that as the integration element, ds=dxdy, moves around the aperture in figure 3 and the distances r_1 and r_2 along with the direction cosines vary.

2.3 Mathematical Analysis, Part 3—The Fresnel Approximation

Figure 3, which is a global coordinate system in x-y-z variables, is modified by adding a local coordinate system in the single dimension of ξ. This is shown in figure 4.

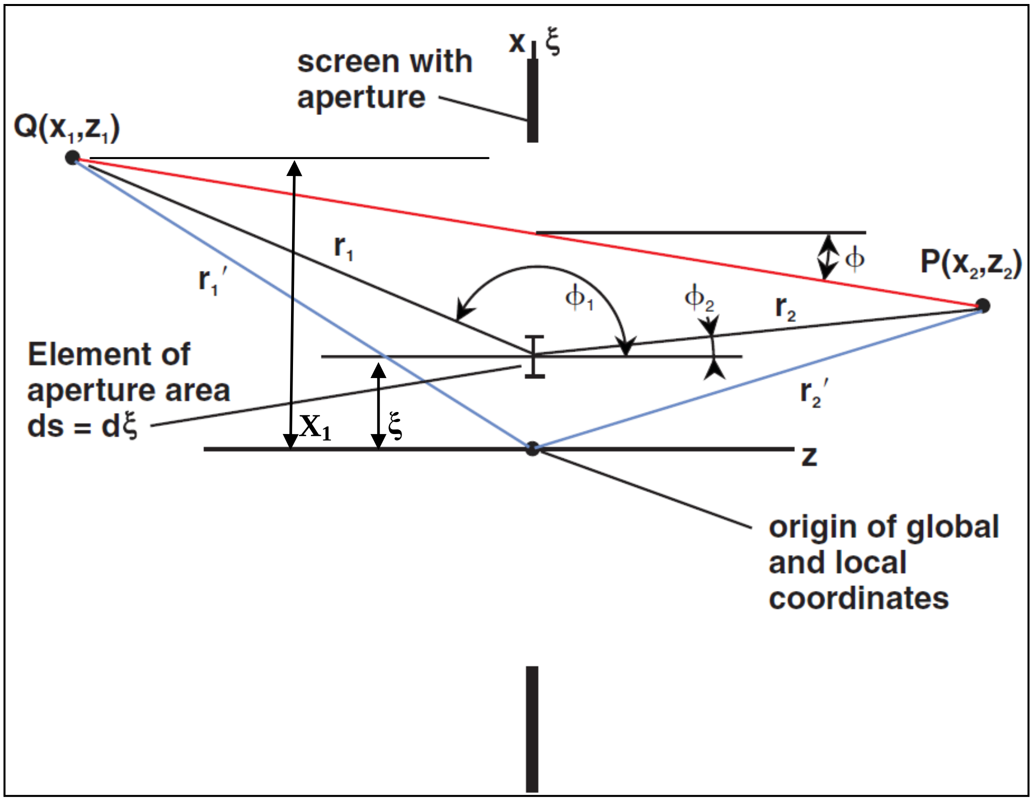

Figure 4. The local coordinate system added to the aperture of figure 3.

The Fresnel approximation further assumes that points Q and P are located a distance from the aperture that is larger than the aperture size. This means that the points are "far" away and the aperture is "small." The factor $1/r_1r_2$ that appears in the integral of equation 19 can now be taken outside of the integral and replace by $1/r_1'r_2'$. The primed values show that the values are measured with respect to the local coordinate system. Since the points are "far" from the aperture, the directions cosines not vary much and can be considered a constant, so the trigonometric equality may be used to replace $(\cos\varphi_1 - \cos\varphi_2)$ with $2\cos\varphi$. These approximations remove certain terms from equation 19, because they are constant. However, the values in the exponent of equation 19 cannot be treated as constants, because the exponent oscillates as the integration element moves over the aperture. With these assumptions and considerations in mind, equation 19 is simplified to

$$U_P = \frac{-Ai}{\lambda}\frac{\cos\phi}{r_1' r_2'}\int_{Aperture} e^{ik(r_1+r_2)} d\xi \tag{20}$$

The Pythagorean and binomial theorems are now used to simplify the integrand (*2*).

$$r_1 = \left[(x_1 - \xi)^2 + z_1^2\right]^{1/2} = z_1\left(1 + \underbrace{\frac{x_1^2}{z_1^2} + \frac{\xi^2}{z_1^2} - \frac{2\xi x_1}{z_1^2}}_{\varepsilon}\right)^{1/2} \tag{21}$$

Because points Q and P are far from the aperture, the three terms in the parenthesis of equation 21 are small, and the binomial theorem may be used. Equation 21 then becomes

$$r_1 = z_1 + \frac{x_1^2}{2z_1} + \frac{\xi^2}{2z_1} - \frac{x_1\xi}{z_1} \tag{22}$$

Substituting equation 22 and the corresponding value for r_2 into equation 20 yield

$$U_P = \frac{-Ai}{\lambda}\frac{\cos\phi}{r_1'r_2'}e^{ik(z_1+z_2)}e^{\frac{ik}{2}\left(\frac{x_1^2}{z_1}+\frac{x_2^2}{z_2}\right)}\int_{Aperture} e^{ik\frac{\xi^2}{2}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)}e^{-ik\left(\frac{x_1\xi}{z_1}+\frac{x_2\xi}{z_2}\right)}d\xi \tag{23}$$

This result is called the Fresnel approximation to the Kirchhoff diffraction integral. A final simplification is required to make the result useful for most practical applications.

2.4 Mathematical Analysis, Part 4—The Fraunhofer Approximation

The central point of the Fraunhofer approximation is to make the first exponential term in equation 23 nearly equal to one. This is the case if ξ is small compared to the distances of the Illumination source, Q, and the illuminated target, P, which are at distances of z_1 and z_2, respectively. This assumptions change equation 23 to

$$U_P = \underbrace{\frac{-Ai}{\lambda}\frac{\cos\phi}{r_1'r_2'}e^{ik(z_1+z_2)}}_{\text{Illumination at Aperture}}\underbrace{e^{\frac{ik}{2}\left(\frac{x_1^2}{z_1}+\frac{x_2^2}{z_2}\right)}}_{\text{Spherical Wave}}\underbrace{\int_{Aperture} e^{-ik\left(\frac{x_1\xi}{z_1}+\frac{x_2\xi}{z_2}\right)}d\xi}_{\text{Fourier Transform}} \tag{24}$$

3. Conclusion

Equation 24 is the basis upon which ARL's holographic interferometry efforts are built, and this is the reason such a long discourse is devoted to the subject. The final result of equation 24 shows that diffraction at an aperture is really a Fourier transforming process. The optical information of the object at point P has been transformed into the frequency information that comprises a hologram, in holography, at the aperture. The aperture may be a film or a digital recording device.

4. References

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2. Schnars, U.; Jueptner, W. *Digital Holography;* Berlin, Germany: Springer, 2010

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